CHERN CLASSES OF DELIGNE-MUMFORD STACKS AND THEIR COARSE MODULI SPACES

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ABSTRACT. Let X be a complex projective algebraic variety with Gorenstein quotient singularities and $\mathcal{X}$ a smooth Deligne-Mumford stack having X as its coarse moduli space. We show that the CSM class $c^{SM}(X)$ coincides with the pushforward to X of the total Chern class $c(T_{I\mathcal{X}})$ of the inertia stack $I\mathcal{X}$. We also show that the stringy Chern class $c_{str}(X)$ of X, whenever is defined, coincides with the pushforward to X of the total Chern class $c(T_{II\mathcal{X}})$ of the double inertia stack $II\mathcal{X}$. Some consequences concerning stringy/orbifold Hodge numbers are deduced.

1. INTRODUCTION

Let X be a complex projective algebraic variety with Gorenstein quotient singularities. In attempt to associate invariants to X, there are at least two possible approaches: one can either view X as a *singular* variety by itself or view X as the coarse moduli space of a *smooth* Deligne-Mumford stack $\mathcal{X}$. Viewing as a singular variety we have the CSM class $c^{SM}(X)$ naturally associated to X. An important property of CSM class is that its degree is equal to the topological Euler characteristic of X:

$$\chi(X) = \int_X c^{SM}(X).$$

The starting point of this note is the observation that $\chi(X)$ is equal to the Euler characteristic $\chi(I\mathcal{X})$ of the *inertia stack* $I\mathcal{X}$. By Gauss-Bonnet formula for Deligne-Mumford stacks we know that

$$\chi(I\mathcal{X}) = \int_{I\mathcal{X}} c_{top}(T_{I\mathcal{X}}),$$

where $c_{top}(T_{I\mathcal{X}})$ is the top Chern class of the tangent bundle $T_{I\mathcal{X}}$ of the inertia stack $I\mathcal{X}$. Therefore the degrees of $c^{SM}(X)$ and $c(T_{I\mathcal{X}})$ are the same. The first result of this note, Theorem 3.4, says that this equality in fact holds for classes themselves: the pushforward to X of $c(T_{I\mathcal{X}})$ is equal to $c^{SM}(X)$. This is a simple consequence of a comparison of the characteristic functions $\mathbf{1}_X, \mathbf{1}_{I\mathcal{X}}$ and MacPherson's natural transformation for Deligne-Mumford stacks [10].

Under some natural assumption on the singularities of X, such as being log terminal, one can define ([5], [3]) the *stringy Chern class* $c_{str}(X)$ for X. In view of the above, it is reasonable to hope that $c_{str}(X)$ can be expressed using some kind of Chern class for the stack $\mathcal{X}$. We show (Theorem 4.1) that $c_{str}(X)$ is equal to the pushforward to X of the total Chern class of the tangent bundle of the *double inertia stack* $II\mathcal{X}$ of $\mathcal{X}$. This implies some formulas for stringy Hodge numbers.

It is known (see e.g. [12]) that if $\mathcal{X}$ and X are *K-equivalent*, i.e. the natural map $\pi : \mathcal{X} \to X$ is birational and we have $K_{\mathcal{X}} = \pi^* K_X$, then stringy Hodge numbers of X coincide with orbifold

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Hodge numbers of $\mathcal{X}$. Together with the above results we find some formulas for orbifold Hodge numbers. In particular we prove in Proposition 4.6 a conjecture in [7].

ACKNOWLEDGMENT

2. PRELIMINARIES

We work over $\mathbb{C}$. Throughout the paper we will make the following assumption.

Assumption 2.1. *$\mathcal{X}$ is a smooth separated Deligne-Mumford stack of finite type over $\mathbb{C}$. Its coarse moduli space X is a projective variety of finite type over $\mathbb{C}$. The structure map is denoted by $\pi : \mathcal{X} \to X$.*

From the scheme theory perspective, Assumption 2.1 means that X is a projective variety of finite type with quotient singularities, and $\mathcal{X}$ is a smooth separated Deligne-Mumford stack having the (singular) variety X as its coarse moduli space.

For an $\mathcal{X}$ as in Assumption 2.1 let $T_{\mathcal{X}}$ be the tangent bundle of $\mathcal{X}$. By definition $T_{\mathcal{X}}$ is a vector bundle over $\mathcal{X}$. As a locally free sheaf, $T_{\mathcal{X}}$ is defined to be the dual of the sheaf $\Omega^1_{\mathcal{X}}$ of differentials on $\mathcal{X}$. See e.g. [11], 7.20 (ii) for the definition of $\Omega^1_{\mathcal{X}}$. The paper [11] also constructs the theory of Chow groups (with rational coefficients) for Deligne-Mumford stacks. In particular the theory of Chern classes is constructed there. Given a vector bundle $\mathcal{V}$ over $\mathcal{X}$, we have the total Chern class of $\mathcal{V}$ which we denote by $c(\mathcal{V})$. The class $c(\mathcal{V})$ belongs to $A^*(\mathcal{X})_{\mathbb{Q}}$, the Chow group of $\mathcal{X}$ with $\mathbb{Q}$-coefficients. In particular, we write $c(T_{\mathcal{X}}) \in A^*(\mathcal{X})_{\mathbb{Q}}$ for the total Chern class of the tangent bundle $T_{\mathcal{X}}$.

Consider a Deligne-Mumford stack of the form[1] $\mathcal{X} = [U/G]$ where U is a smooth scheme and G is a linear algebraic group. The paper [6] constructs a theory of equivariant Chow groups (with integer coefficients) for the G-action on U, denoted by $A^G_*(U)$. By [6], Proposition 14, we have

$$A^G_*(U) \otimes \mathbb{Q} = A^{\dim U - *}(\mathcal{X})_{\mathbb{Q}}.$$

The tangent bundle T_U is a G-equivariant vector bundle on U. The construction of [6], Section 2.4 associates to T_U its equivariant total Chern classes $c^G(T_U) \in A^G_*(U) \otimes \mathbb{Q}$. Under the above identification of Chow groups, we have $c^G(T_U) = c(T_{\mathcal{X}})$.

Remark 2.2. We may consider the equivariant Chern class $c^G(T_U)$ as a class in the equivariant cohomology $H^*_G(U) \otimes \mathbb{Q}$ by using the cycle map.

In the paper we make heavy use of the theory of constructible functions on Deligne-Mumford stacks. Our reference for this is [8], to which we refer the readers for a detailed treatment of of this. Below we recall some aspects of the theory.

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[1]Stacks of this form are called quotient stacks.

Let $\mathcal{X}$ be a Deligne-Mumford stack as in Assumption 2.1. Denote by $\mathcal{X}(\mathbb{C})$ the set of $\mathbb{C}$-points of $\mathcal{X}$. By [8], Definition 4.1, a subset of $\mathcal{X}(\mathbb{C})$ is *constructible* if it is a finite union of sets of the form $\mathcal{X}_i(\mathbb{C})$ where each $\mathcal{X}_i$ is a finite type substack of $\mathcal{X}$. A function $\phi : \mathcal{X}(\mathbb{C}) \to \mathbb{Q}$ is called *constructible* if $\phi(\mathcal{X}(\mathbb{C}))$ is finite and $\phi^{-1}(c) \subset \mathcal{X}(\mathbb{C})$ is constructible for any $c \in \phi(\mathcal{X}(\mathbb{C})) \setminus \{0\}$, see [8], Definition 4.3. Denote by $CF(\mathcal{X})$ the group of constructible functions on $\mathcal{X}$. For a constructible set $C \subset \mathcal{X}(\mathbb{C})$ define its *characteristic function* $\mathbf{1}_C : \mathcal{X}(\mathbb{C}) \to \mathbb{Q}$ by

$$\mathbf{1}_C(c) = \begin{cases} 1 & \text{if } c \in C, \\ 0 & \text{if } c \notin C. \end{cases}$$

Clearly $\mathbf{1}_C$ is constructible, and $CF(\mathcal{X})$ is additively generated by characteristic functions. Define the function $\mathbf{1}_{\mathcal{X}} : \mathcal{X}(\mathbb{C}) \to \mathbb{Q}$ to be $\mathbf{1}_{\mathcal{X}} := \mathbf{1}_{\mathcal{X}(\mathbb{C})}$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of Deligne-Mumford stacks. In [8], Definition 4.17 (a), the notion of "stack pushforward" by f is defined. This notion of pushforward, which we simply call *pushforward* and denote by f_*, will be used in this paper. We recall its definition as follows. Define a function $e_{\mathcal{X}} : \mathcal{X}(\mathbb{C}) \to \mathbb{Q}$ by $e_{\mathcal{X}}(c) := |G_c|$, where $|G_c|$ is the order of the isotropy group G_c at the point $c \in \mathcal{X}(\mathbb{C})$. A function $e_{\mathcal{Y}} : \mathcal{Y}(\mathbb{C}) \to \mathbb{Q}$ is similarly defined. Let $\phi : \mathcal{X}(\mathbb{C}) \to \mathbb{Q}$ be a constructible function. Define $f_*\phi : \mathcal{Y}(\mathbb{C}) \to \mathbb{Q}$ by

$$(2.1) \qquad f_*\phi(t) := e_{\mathcal{Y}}(t)\chi(\mathcal{X}(\mathbb{C}), \frac{1}{e_{\mathcal{X}}}\phi\mathbf{1}_{f^{-1}(t)}), \quad t \in \mathcal{Y}(\mathbb{C}),$$

where $\chi(-, -)$ is the weighted Euler characteristic as in [8], Definition 4.8. As pointed out in [8], page 599, since we work with Deligne-Mumford stacks, the pushforward f_* is always defined. By [8], Corollary 4.14, the pushforard satisfies functoriality: $(f \circ g)_* = f_*g_*$.

Lemma 2.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite proper representable étale morphism of Deligne-Mumford stacks. Then the following equality of constructible functions hold: $f_*\mathbf{1}_{\mathcal{X}} = (\deg f)\mathbf{1}_{f(\mathcal{X})}$.*

Proof. For a geometric point $y \in f(\mathcal{X}(\mathbb{C}))$ we may write $f^{-1}(y) = \cup_{i \in I}x_i$, where $x_i \in \mathcal{X}(\mathbb{C})$ and I is a finite set. We have $f_*\mathbf{1}_{\mathcal{X}}(y) = e_{\mathcal{Y}}(y)\sum_{i \in I}\frac{1}{e_{\mathcal{X}}(x_i)}$, which is equal to $\deg f$. $\qquad\square$

Consider again a Deligne-Mumford quotient stack $\mathcal{X} = [U/G]$ where U is a smooth scheme and G is a linear algebraic group. In [10] the notion of constructible functions for this kind of stacks is also defined. By definition (see [10], Section 3.4) a constructible function on $\mathcal{X}$ is a G-invariant constructible function on U. Let $CF_{inv}^G(U)$ be the group[2] of G-invariant constructible functions on U. By [10], Lemma 3.3, this group is independent of the presentation of $\mathcal{X}$ as a quotient. Given a finite type substack $\mathcal{Z} \subset \mathcal{X}$, define $U_{\mathcal{Z}} := \mathcal{Z} \times_{\mathcal{X}} U \subset U$. It is easy to see that the map $\mathbf{1}_{\mathcal{Z}(\mathbb{C})} \mapsto \mathbf{1}_{U_{\mathcal{Z}}(\mathbb{C})}$ defines an isomorphism

$$(2.2) \qquad\qquad CF(\mathcal{X}) \simeq CF_{inv}^G(U).$$

It is also straightforward to check that under the identification above, the notion of pushforward for $CF_{inv}^G(U)$, as defined in [10], Section 2.6, coincides with the pushforward in (2.1). Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of Deligne-Mumford stacks. Denote by $f_* : CF(\mathcal{X}) \to CF(\mathcal{Y})$ the pushforward as in (2.1), and by $f_{*'} : CF(\mathcal{X}) \to CF(\mathcal{Y})$ the pushforward in [10], Section 2.6 after the identification (2.2). By construction we also have functorialty property $(f \circ g)_{*'} = f_{*'}g_{*'}$. Given $\phi \in CF(\mathcal{X})$, to compare $f_*\phi$ and $f_{*'}\phi$ it suffices to compare them pointwise. Therefore we may

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[2]In [10] this group is denoted by $\mathcal{F}_{inv}^G(U)$.

assume that f is of the form $f : BG \to BH$ given by a homomorphism $G \to H$ of finite groups, and $\phi = \mathbf{1}_{BG}$.

In case G is trivial, i.e. $f : \text{pt} \to BH$, we have $f_* \mathbf{1}_{\text{pt}} = |H| \mathbf{1}_{BH}$ by Lemma 2.3. Let H acts on itself by translations. Then we may present the map f as a quotient by H of the constant map $\tilde{f} : H \to \text{pt}$. Then it follows from the definitions that $f_{*'} \mathbf{1}_{\text{pt}} = \tilde{f}_* \mathbf{1}_H = |H| \mathbf{1}_{\text{pt}} = |H| \mathbf{1}_{BH}$. Thus $f_* = f_{*'}$ in this case.

Suppose G is not necessarily trivial. Let $p : \text{pt} \to BG$ be an atlas of BG. Then we have $p_* \mathbf{1}_{\text{pt}} = |G| \mathbf{1}_{BG} = p_{*'} \mathbf{1}_{\text{pt}}$ and $(f \circ p)_* \mathbf{1}_{\text{pt}} = |H| \mathbf{1}_{BH} = (f \circ p)_{*'} \mathbf{1}_{\text{pt}}$ by the case above. Thus by functoriality of pushforward, we have

$$f_* \mathbf{1}_{BG} = \frac{1}{|G|} f_* p_* \mathbf{1}_{\text{pt}} = \frac{1}{|G|} (f \circ p)_* \mathbf{1}_{\text{pt}} = \frac{1}{|G|} (f \circ p)_{*'} \mathbf{1}_{\text{pt}} = \frac{1}{|G|} f_{*'} p_{*'} \mathbf{1}_{\text{pt}} = f_{*'} \mathbf{1}_{BG},$$

which is what we want.

3. Chern classes

Let $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ be the diagonal morphism. Recall that the *inertia stack* of $\mathcal{X}$ is defined to be $I\mathcal{X} := \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$, see for example [5], Definition 5.1. and Lemma 5.2. Let $p : I\mathcal{X} \to \mathcal{X}$ be the natural projection.

If $\mathcal{X} = [M/G]$ where M is a scheme and G is a finite group, then the inertia stack can be described as follows:

$$I[M/G] = \coprod_{(g):\text{conjugacy class of } G} [M^g / C_G(g)].$$

See for example [5], Lemma 5.6 for a proof of this fact.

The following is well-known.

Lemma 3.1 (c.f. [2], Lemma 2.2.3). *Let $\mathcal{X}$ be a separated Deligne-Mumford stack, and X its coarse moduli space. There is an étale covering $\coprod_a X_a \to X$ such that for each a there is a scheme U_a and a finite group G_a acting on U_a, such that $\mathcal{X} \times_X X_a \simeq [U_a / G_a]$.*

Proposition 3.2. *Let $\mathcal{X}$ be as in Assumption 2.1. Then*

$$\pi_* p_* \mathbf{1}_{I\mathcal{X}} = \mathbf{1}_X.$$

Proof. The question is local on X. Lemma 2.3 allows one to replace X by an étale cover. In view of Lemma 3.1, we are reduced to the case $\mathcal{X} = [M/G]$ where M is a scheme and G is a finite group. Denote by $\rho : M \to [M/G]$ the atlas map, and $\pi : [M/G] \to M/G$ the map to coarse moduli scheme.

Put $\alpha := \frac{1}{|G|} \sum_{g \in G} \mathbf{1}_{M^g}$. For a geometric point $x \in M$ we denote by $[x]$ the corresponding geometric point in M/G. The calculation in the proof of [10], Proposition 6.1 gives

$$(\pi \circ \rho)_* \alpha([x]) = \frac{1}{|G|} \sum_{g \in G} (\pi \circ \rho)_* \mathbf{1}_{M^g}([x])$$

$$= \frac{1}{|G|} \sum_{x' \in G.x} \sum_{g \in G} \mathbf{1}_{M^g}(x')$$

$$= \frac{1}{|G|} |G.x| |\mathrm{Stab}_x(G)| = 1.$$

Thus $(\pi \circ \rho)_* \alpha = \mathbf{1}_{M/G}$. It is easy to see that

$$\frac{1}{|G|} \sum_{g \in G} \mathbf{1}_{M^g} = \sum_{(g): \text{conjugacy class}} \frac{\mathbf{1}_{M^g}}{|C_G(g)|}.$$

By Lemma 2.3, we see that the pushforward of $\mathbf{1}_{M^g}$ to $[M^g/C_G(g)]$ is equal to $|C_G(g)| \mathbf{1}_{[M^g/C_G(g)]}$. Therefore the pushforward of α to $I[M/G]$ is equal to $\mathbf{1}_{I[M/G]}$. The result follows. $\square$

Remark 3.3. In the proof of Proposition 3.2 one can argue without using Lemmas 3.1 and 2.3, as follows: Let $x : \mathrm{Spec}\, k \to X$ be a geometric point. Then $\mathrm{Spec}\, k \times_{x,X,\pi} \mathcal{X}$ is isomorphic to BG for some finite group G. Moreover, we have

$$\mathrm{Spec}\, k \times_{x,X,\pi \circ p} I\mathcal{X} \simeq \coprod_{(g): \text{conjugacy class of } G} [\mathrm{Spec}\, k/C_G(g)].$$

We conclude by using the equality

$$\sum_{(g): \text{conjugacy class of } G} \frac{1}{|C_G(g)|} = 1.$$

Theorem 3.4. *Let $\mathcal{X}$ be as in Assumption 2.1. Then*

$$\pi_* p_* c(T_{I\mathcal{X}}) = c^{SM}(X).$$

Proof. Assumption 2.1 on $\mathcal{X}$ implies that $I\mathcal{X}$ is a quotient stack $[W/H]$ of a quasi-projecitve scheme W by a linear algebraic group H (see [9], Theorem 4.4). This allows us to apply [10], Theorem 3.5 to Proposition 3.2 to obtain

$$\pi_* p_* C_*(\mathbf{1}_{I\mathcal{X}}) = c^{SM}(X) \cap [X].$$

The function $\mathbf{1}_{I\mathcal{X}}$ is identified with $\mathbf{1}_W$ under the identification $CF(I\mathcal{X}) \simeq CF^H_{inv}(W)$ of groups of constructible functions. This implies that $C_*(\mathbf{1}_{I\mathcal{X}}) = C^H_*(\mathbf{1}_W)$. The normalization property of C^H_* implies that $C^H_*(\mathbf{1}_W) = c^H(T_W) \cap [W]_H$. Since the H-equivariant Chern class $c^H(T_W)$ is identified with the Chern class $c(T_{I\mathcal{X}})$ under the identification $H^*_H(W) \simeq H^*(I\mathcal{X})$, the result follows. $\square$

4. STRINGY CHERN CLASSES

Let IX be the coarse moduli scheme of the inertia stack $I\mathcal{X}$. There is a diagram

$$
\begin{array}{ccc}
I\mathcal{X} & \xrightarrow{\ p\ } & \mathcal{X} \\
\tilde{\pi} \downarrow & & \pi \downarrow \\
IX & \xrightarrow{\ \bar{p}\ } & X.
\end{array}
$$

In [5] the authors define a constructible function Φ_X and the *stringy Chern class* of X:

$$
c_{str}(X) := c^{SM}(\Phi_X).
$$

Theorem 4.1. *Let $\mathcal{X}$ be as in Assumption 2.1. Then*

(1) $\Phi_X = \bar{p}_* \mathbf{1}_{IX}$.
(2) $c_{str}(X) = \pi_* q_* c(T_{II\mathcal{X}})$, *where* $q : II\mathcal{X} \to \mathcal{X}$ *is the natural projection from the* double *inertia stack* $II\mathcal{X}$ *to* $\mathcal{X}$.

Proof. Let $g : W \to V$ be a resolution of singularity, $f : V' \to V$ an étale map, $W' := W \times_V V'$, and $f' : W' \to W$, $g' : W' \to V'$ the natural projections. Then g' is also a resolution of singularity, and we have

$$
\begin{aligned}
f'^* K_{W/V} &= f'^* K_W - f'^* g^* K_V \\
&= K_{W'} - g'^* f^* K_V \quad \text{(since } f' \text{ is étale)} \\
&= K_{W'} - g'^* K_{V'},
\end{aligned}
$$

where the last step is justified as follows: Let $j : U \to V$ be the smooth locus of V, $U' := U \times_V V'$, and $j' : U' \to V'$, $f_U : U' \to U$ the natural projections. Then we have $K_V := j_*(\wedge^{\dim U} \Omega^1_U)$. Thus

$$
f^* K_V = f^* j_*(\wedge^{\dim U} \Omega^1_U) = j'_* f_U^*(\wedge^{\dim U} \Omega^1_U) = j'_*(\wedge^{\dim U'} \Omega^1_{U'}) = K_{V'}.
$$

It follows that $f'^* K_{W/V} = K_{W'/V'}$. Applying [5], Proposition 2.3, we see that part (1) can be checked on an étale covering of X. Therefore by Lemma 3.1 we may assume that $X = M/G$ for some scheme M and finite group G. In this case part (1) is [5], Theorem 6.1.

Since the inertia stack of $I\mathcal{X}$ is by definition the double inertia stack $II\mathcal{X}$, part (2) follows immediately from Theorem 3.4 applied to IX. $\qquad\square$

Let $e_{str}(X)$ be the stringy Euler characteristic of X. By [5], Proposition 4.4, we have $e_{str}(X) = \int_X c_{str}(X)$. The following is immediate from Theorem 4.1.

Corollary 4.2. $e_{str}(X) = \int_{II\mathcal{X}} c_{top}(T_{II\mathcal{X}})$.

Let $n = \dim X$. In [4], Definition 3.1, Batyrev defined a number $c_{st}^{1,n-1}(X)$, which can be interpreted as a stringy analogue of the Chern number $c_1(X) c_{top-1}(X)$. Properties of $c_{str}(X)$ (see the proof of [5], Proposition 4.4) implies that

$$
c_{st}^{1,n-1}(X) = \int_X c_1(X) c_{str}(X).
$$

Theorem 4.1 implies

Corollary 4.3.
$$c_{st}^{1,n-1}(X) = \int_{II\mathcal{X}} q^*\pi^*c_1(X)c_{top-1}(T_{II\mathcal{X}}).$$

Remark 4.4. This proves a more general form of [7], Conjecture A.2.

Under additional hypotheses we can deduce some consequences on orbifold Hodge numbers.

Assumption 4.5. *Let $\mathcal{X}$ be as in Assumption 2.1. In addition X is assumed to be Gorenstein, the map $\pi : \mathcal{X} \to X$ is assumed to be birational, and $K_{\mathcal{X}} = \pi^*K_X$.*

Let $I\mathcal{X} = \coprod_{i\in\mathcal{I}} \mathcal{X}_i$ be the decomposition into disjoint union of connected components. For each $\mathcal{X}_i$ one can associate a rational number $age(\mathcal{X}_i)$ called the *age* of $\mathcal{X}_i$. See for example [12] for a definition.

The numbers $age(\mathcal{X}_i)$, which arise naturally in the context of Riemann-Roch formula for twisted curves (see [1], Section 7.2), are relevant to us due to their presence in the *Chen-Ruan orbifold cohomology*. By definition, the Chen-Ruan cohomology groups of $\mathcal{X}$ are $H^*_{CR}(\mathcal{X}, \mathbb{C}) := H^*(I\mathcal{X}, \mathbb{C}) = \oplus_{i\in\mathcal{I}} H^*(\mathcal{X}_i, \mathbb{C})$. The numbers $age(\mathcal{X}_i)$ are used to define a new grading on $H^*_{CR}(\mathcal{X}, \mathbb{C})$:
$$H^p_{CR}(\mathcal{X}, \mathbb{C}) := \oplus_{i\in\mathcal{I}} H^{p-2age(\mathcal{X}_i)}(\mathcal{X}_i, \mathbb{C}).$$
The Dolbeault cohomology version of this can be similarly defined:
$$H^{p,q}_{CR}(\mathcal{X}, \mathbb{C}) := \oplus_{i\in\mathcal{I}} H^{p-age(\mathcal{X}_i),q-age(\mathcal{X}_i)}(\mathcal{X}_i, \mathbb{C}).$$

Proposition 4.6. *Let $\mathcal{X}$ be as in Assumption 4.5. Then the following holds.*

(4.1)
$$c_{st}^{1,n-1}(X) = \int_{II\mathcal{X}} q^*c_1(\mathcal{X})c_{top-1}(T_{II\mathcal{X}}).$$

(4.2)
$$\sum_{i\in\mathcal{I}}\sum_{p\geq 0}(-1)^p \left(p + age(\mathcal{X}_i) - \frac{dim\mathcal{X}}{2}\right)^2 \chi(\mathcal{X}_i, \Omega^p_{\mathcal{X}_i}) = \frac{1}{12}\int_{II\mathcal{X}} dim\mathcal{X} c_{top}(T_{II\mathcal{X}}) + 2c_1(T_{\mathcal{X}})c_{top-1}(T_{II\mathcal{X}}).$$

Proof. (4.1) follows from $K_{\mathcal{X}} = \pi^*K_X$.

We now prove (4.2). Under Assumption 4.5 a result of T. Yasuda [12] asserts that Batyrev's *stringy Hodge numbers* $h_{st}^{p,q}(X)$ coincide with *orbifold Hodge numbers* $h_{orb}^{p,q}(\mathcal{X}) := \dim H^{p,q}_{CR}(\mathcal{X}, \mathbb{C})$. We refer to [12] for relevant definitions. In terms of generating functions, we have
$$E_{st}(X, s, t) = E_{orb}(\mathcal{X}, s, t),$$
where
$$E_{st}(X, s, t) := \sum_{p,q\geq 0}(-1)^{p+q}h_{st}^{p,q}(X)s^p t^q$$
is the stringy E-polynomial and
$$E_{orb}(\mathcal{X}, s, t) := \sum_{p,q\geq 0}(-1)^{p+q}h_{orb}^{p,q}(\mathcal{X})s^p t^q$$
is the orbifold E-polynomial. Combining this with Corollary 3.10 of [4] we find that

(4.3)
$$\sum_{p,q}(-1)^{p+q}\left(p - \frac{dim\mathcal{X}}{2}\right)^2 h_{orb}^{p,q}(\mathcal{X}) = \frac{dim X}{12}e_{str}(X) + \frac{1}{6}c_{st}^{1,n-1}(X).$$

We rewrite the left-hand side as follows. By definition of $h_{orb}^{p,q}(\mathcal{X})$,

$$\sum_{p,q}(-1)^{p+q}\left(p-\frac{\dim\mathcal{X}}{2}\right)^2 h_{orb}^{p,q}(\mathcal{X})$$

$$=\sum_{i\in\mathcal{I}}\sum_{p,q}(-1)^{p+q}\left(p-\frac{\dim\mathcal{X}}{2}\right)^2 \dim H^{p-age(\mathcal{X}_i),q-age(\mathcal{X}_i)}(\mathcal{X}_i,\mathbb{C})$$

$$=\sum_{i\in\mathcal{I}}\sum_{p,q\geq 0}(-1)^{p+q+2age(\mathcal{X}_i)}\left(p+age(\mathcal{X}_i)-\frac{\dim\mathcal{X}}{2}\right)^2 \dim H^{p,q}(\mathcal{X}_i,\mathbb{C})\quad\text{(re-indexing)}$$

$$=\sum_{i\in\mathcal{I}}\sum_{p\geq 0}(-1)^{p}\left(p+age(\mathcal{X}_i)-\frac{\dim\mathcal{X}}{2}\right)^2 \left(\sum_{q\geq 0}(-1)^{q}\dim H^{p,q}(\mathcal{X}_i,\mathbb{C})\right).$$

In the last equality we used the fact that $age(\mathcal{X}_i)\in\mathbb{Z}$, which is true because X is Gorenstein. Thus we arrive at

$$(4.4)\qquad \sum_{i\in\mathcal{I}}\sum_{p\geq 0}(-1)^{p}\left(p+age(\mathcal{X}_i)-\frac{\dim\mathcal{X}}{2}\right)^2 \chi(\mathcal{X}_i,\Omega_{\mathcal{X}_i}^p)=\frac{\dim X}{12}e_{str}(X)+\frac{1}{6}c_{st}^{1,n-1}(X).$$

(4.2) now follows from Corollary 4.2 and (4.1). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.7. (4.2) is conjectured to hold for any smooth proper Deligne-Mumford stack with projective coarse moduli space, see [7], Conjecture 3.2'.

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